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A MATHEMATICAL MODEL OF GPS SATELLITE TIME AND FREQUENCY BASED ON THE SPECIAL THEORY OF RELATIVITY

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ABSTRACT
The Global Positioning System (GPS) relies on highly precise atomic clocks in satellites and ground stations to determine global time and position accurately. Relativistic effects, including gravitational fields and satellite orbital motion, induce frequency shifts in these clocks, necessitating corrections for accurate navigation. This study developed a novel approach to integrate general coordinates into GPS, emphasizing implementation simplicity. We investigated the application of special relativity to calculate time and frequency shifts in GPS satellites. Our analysis estimated a daily relativistic correction of approximately 21.6 microseconds, with a positional imprecision of 6.48 km per day if uncorrected. By employing a general time-space coordinate system, we calculated position-4 and velocity-4 vectors, accounting for special relativistic effects on GPS time and frequency. The proposed method simplifies relativistic corrections while enhancing their conceptual clarity. Results demonstrate improved precision and efficiency in GPS positioning through the integration of general coordinates with a streamlined approach. This research offers valuable insights for international engineering communities, scholars, and practitioners, advancing the understanding of relativity's impact on GPS satellite operations.
Keywords: *GPS, Special relativity theory, mathematical model, time, and frequency.*

1. INTRODUCTION

The Global Positioning System (GPS) exemplifies the application of modern physics, particularly Einstein's theories of special and general relativity, to advanced technology in the digital era. Special relativity, proposed in 1905, introduced a four-dimensional space-time framework through rigorous tensor analysis, challenging the notion of absolute space and time (Fathollahi, 2023). General relativity, published in 1916, extended this framework to high-speed conditions, incorporating energy-momentum tensors and fundamental space-time frames (Delva,2018). These theories underpin modern physics and are integral to university curricula in physics and engineering, where understanding Riemannian geometry is essential for grasping relativistic principles (Abbott et al., 2016; Alvager, Nilsson, & Kjellman, 1963; Ashby, 2003; Chambat, 2001; Mewes, 2018; Rindler, 2006; Shapiro, Pettengill, & Michael, 1968; Singer, 1960; Zus et al., 2012). The GPS relies on precise atomic clocks in satellites and on Earth to determine global time and position. However, gravitational fields and satellite orbital motion induce significant frequency

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shifts, necessitating relativistic corrections for accurate navigation (Aparicio & Laroche, 2011;Santos, 2007). Without accounting for these effects, GPS functionality would be compromised. This study investigated the implications of special relativity for GPS navigation, aiming to enhance the understanding of students and practitioners. Previous research has acknowledged relativis .

Tactic effects on GPS for over five decades, yet challenges persist in simplifying computational models. We developed a novel approach utilizing a general space-time coordinate system to calculate position-4 and velocity-4 vectors, addressing the impact of special relativity on GPS time and frequency. By applying special relativity principles, this work simplified relativistic corrections and improved the clarity of GPS positioning models. These advancements offer potential for more efficient navigation systems and provide a foundation for future refinements in global positioning technology.

2. RESEARCH METHODS

By using atomic clocks on satellites as well as on the ground, the GPS is capable of determining time and position anywhere in the world. The gravitational field and satellite orbital motion cause such a significant frequency shift in this atomic clock that the system cannot operate appropriately if relativistic influences are ignored. Using the special theory of relativity, we investigate how GPS satellites moving with a certain velocity can be determined both at a specific frequency and at a specific time. The study discusses the use of special relativity to navigate using GPS systems, which use a generalized coordinate system based on position-4 vectors. To validate our findings, we analyzed data from (Santos, 2007).

3. RESULTS AND DISCUSSIONS

This section explains the derivation formula for vector-4 position, vector-4 velocity, and time dilation. In the case of the GPS satellite moving at a motion frame speed V, it is located at a height h above the surface of the Earth (**Fig.1**).

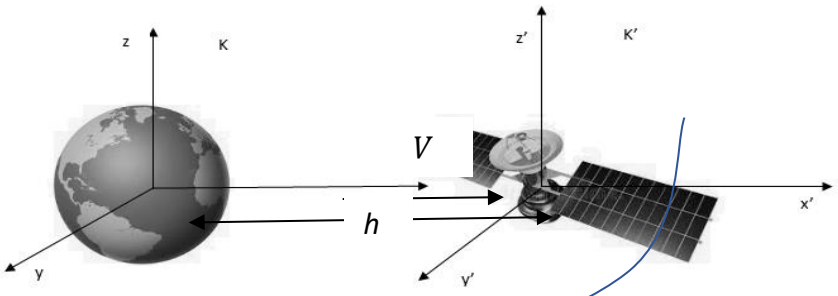


Fig. 1. GPS satellite orbits around the world.

We use the 4-position vector using a Cartesian space coordinate system to gain a better understanding of the general coordinate frame. Assume that the GPS satellite moved along the positive x-axis (x^1) in the motion frame K' for a short distance, as shown in **Fig. 2.**

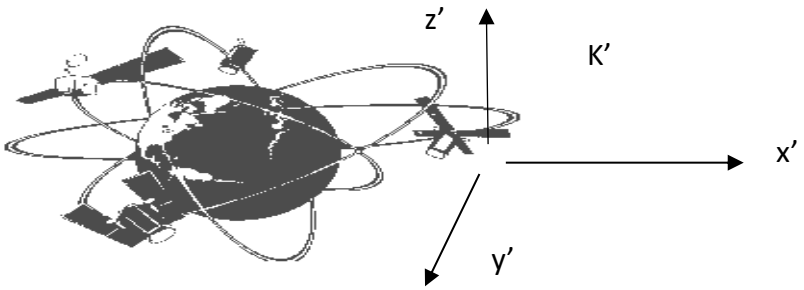


Fig. 2. GPS satellites moving in the x-axis direction.

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According to the Lorentz transformation for the 4-position vector of the stationary frame, *K* is (*ct,x,y,z*) in the Cartesian time and space frame, or (ct, r, θ, φ) in the spherical time and space frame, and in general, we can write (x^0, x^1, x^2, x^3) in a standard time frame and coordinate space. The 4-position vector in the motion frame, *K*', of the GPS satellite is (ct', x', y', z'), $(ct', r', \theta', \varphi')$, or $(x^{0\prime}, x^{1\prime}, x^{2\prime}, x^{3\prime})$. The length of the square element in general time and space coordinate, Spherical coordinate, and Cartesian coordinate for the 4-position vector can be written as follows (Eqs. (1) and (2)

$$S = (x^0, x^1, x^2, x^3) = (ct, r, \theta, \varphi) = (ct, x, y, z) \quad (1)$$

$$\begin{aligned} dS^2 &= -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \\ &= -c^2dt^2 + dr^2 + r^2d\theta^2 + r^2sin^2\theta d\varphi^2 \\ &= -c^2dt^2 + dx^2 + dy^2 + dz^2 \end{aligned} \quad (2)$$

Eqs. (3) to (5) can be written in the K' motion system for solutions in Cartesian space.

$$x' = \gamma(x - Vt); x' = ct', \quad (3)$$

$$y' = y, \quad (4)$$

$$z' = z, \quad (5)$$

For determining t' from Eqs. (6) to (10), we examine the static frame *K*

$$x = \gamma(x' + Vt'); x = ct \quad (6.\ a)$$

$$t = \gamma\left(t' + \frac{V}{c^2}x'\right), \quad (6.\ b)$$

$$x = \gamma(\gamma(x - Vt) + Vt'), \quad (7)$$

$$x = \gamma^2 x - \gamma^2 Vt + \gamma Vt', \quad (8)$$

$$\gamma Vt' = x(1 - \gamma^2) + \gamma^2 Vt, \quad (9)$$

$$t' = \frac{x(1 - \gamma^2)}{\gamma V} + \gamma t, \quad (10)$$

When we return to the motion frame K' and substitute Eq. (10) into (3), we get Eqs. (11) to (14)

$$x' = \gamma(x - Vt), \quad (11)$$

$$ct' = \gamma(x - Vt), \quad (12)$$

$$c\left(\frac{x(1 - \gamma^2)}{\gamma V} + \gamma t\right) = \gamma(x - Vt), \quad (13)$$

$$c\left(\frac{ct(1 - \gamma^2)}{\gamma V} + \gamma t\right) = \gamma(ct - Vt), \quad (14)$$

Eliminate the time *t* in Eq.(14), and we get Eqs. (15) to (18)

$$\left(\frac{c^2(1 - \gamma^2)}{\gamma V}\right) = \gamma(-V), \quad (15)$$

$$c^2(1 - \gamma^2) = -\gamma^2 V^2, \quad (16)$$

$$\gamma^2(c^2 - V^2) = c^2, \quad (17)$$

$$\gamma = \frac{1}{\sqrt{\left(1 - \frac{V^2}{c^2}\right)}}, \quad (18)$$

After that, we obtain the 4-position vector as follows.

$$x' = \gamma(x - Vt), \quad (19)$$

$$x' = \frac{(x - Vt)}{\sqrt{\left(1 - \frac{V^2}{c^2}\right)}}, \quad (20)$$

For constant y and z, we get Eqs. (21) to (22)

$$y' = y \text{ and } z' = z, \quad (21)$$

$$t' = \frac{x'}{c} = \gamma\left(\frac{x}{c} - \frac{Vt}{c}\right) = \gamma\left(t - \frac{Vt}{c}\right) = \gamma\left(t - \frac{Vct}{c^2}\right), \quad (22.a)$$

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$$t' = \gamma\left(t - \frac{Vx}{c^2}\right), \tag{22.b}$$

With a certain position is written as in Eq.(23)

$$ct' = \gamma\left(ct - \frac{Vx}{c}\right), \tag{23}$$

In the matrix form, the 4-position vector can be expressed by Eqs. (24) to (26)

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma V}{c} & 0 & 0 \\ -\gamma V/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \tag{24}$$

$$\begin{pmatrix} \tilde{x}^0 \\ \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma V}{c} & 0 & 0 \\ -\gamma V/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{25}$$

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma V}{c} & 0 & 0 \\ -\gamma V/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} T_0^0 & T_1^0 & 0 & 0 \\ T_0^1 & T_1^1 & 0 & 0 \\ 0 & 0 & T_2^2 & 0 \\ 0 & 0 & 0 & T_3^3 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{26.a}$$

$$x^{i\prime} = T_j^i x^j, \tag{26.b}$$

A contravariant 4-position vector can be obtained from equation (26) using Cartesian coordinates, where $x^2$and x^3 are constants but $x^0$and x^1 are not. In general, it can be written as in Eq. (27)

$$\begin{aligned} x^{0\prime} &= T_j^0 x^j = T_0^0 x^0 + T_1^0 x^1 + T_2^0 x^2 + T_3^0 x^3 = \gamma x^0 - \frac{\gamma V}{c} x^1, \\ x^{1\prime} &= T_j^1 x^j = T_0^1 x^0 + T_1^1 x^1 + T_2^1 x^2 + T_3^1 x^3 = -\frac{\gamma V}{c} x^0 + \gamma x^1, \end{aligned} \tag{27}$$

In case $x^1$and x^3 are constants but x^0 and x^2 are not, and the GPS satellite only moves in x^2 (θ) with speed V as shown in **Fig.3**, then Eq. (27) can be changed into Eq. (28)

$$\begin{aligned} x^{0\prime} &= T_j^0 x^j = T_0^0 x^0 + T_1^0 x^1 + T_2^0 x^2 + T_3^0 x^3 = \gamma x^0 - \frac{\gamma V}{c} x^2, \\ x^{2\prime} &= T_j^2 x^j = T_0^2 x^0 + T_1^2 x^1 + T_2^2 x^2 + T_3^2 x^3 = -\frac{\gamma V}{c} x^0 + \gamma x^2, \end{aligned} \tag{28}$$

Every element of the 4-position vector can be constructed as in Eqs. (29) and (30)

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & -\frac{\gamma V}{c} & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma V/c & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{29}$$

$$\begin{pmatrix} ct' \\ r' \\ r'\theta' \\ r'\sin(\theta)'\varphi' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & -\frac{\gamma V}{c} & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma V/c & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ r \\ r\theta \\ r\sin(\theta)\varphi \end{pmatrix}, \tag{30}$$

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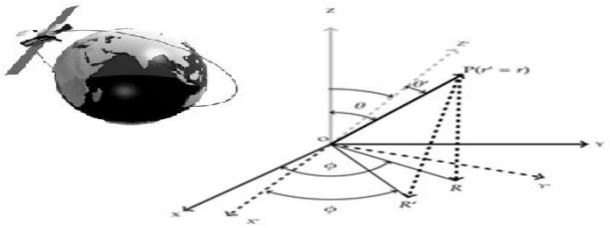


Fig. 3. Change in GPS satellite position as a function of angle θ.

In case x^1 and x^2 are constants but x^0 and x^3 are not, and the GPS satellite only moves in $x^3(\varphi)$ with speed V as shown in **Fig.4**, then Eq. (27) can be changed into Eq. (31)

$$x^{0\prime} = \gamma x^0 - \frac{\gamma V}{c} x^3$$
$$x^{3\prime} = -\frac{\gamma V}{c} x^0 + \gamma x^3 \tag{31}$$

Every element of the 4-position vector can be constructed as in Eqs. (32) and (33)

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\frac{\gamma V}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma V/c & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{32}$$

$$\begin{pmatrix} ct' \\ r' \\ r'\theta' \\ r'\sin(\theta)'\varphi' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\frac{\gamma V}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma V/c & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ r \\ r\theta \\ r\sin(\theta)\varphi \end{pmatrix} \tag{33}$$

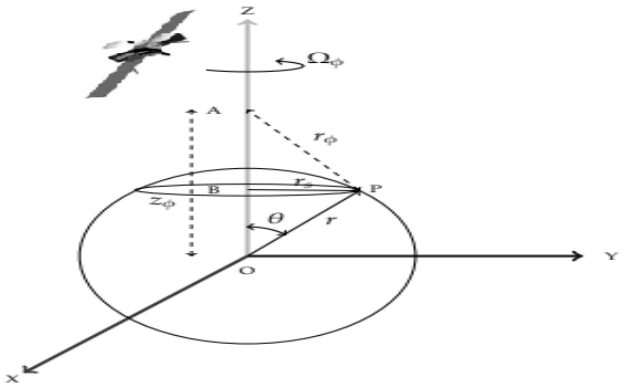


Fig.4. Change in GPS satellite position as a function of angle φ.

Eqs. (26), (29), and (32) show the 4-position vector in general coordinates, which can be Cartesian coordinates or spherical coordinates. In Eq. (6), we can determine that there is time dilation if the initial position is the same in the motion frame x'.

$$\Delta t = \gamma\left(t' + \frac{V}{c^2}x'\right) - \gamma\left(t_o{}' + \frac{V}{c^2}x'\right) = \gamma\Delta t', \tag{34}$$

$$\Delta t = \frac{1}{\sqrt{\left(1 - \frac{V^2}{c^2}\right)}}\Delta t' \approx \left(1 + \frac{1}{2}\frac{V^2}{c^2} + Const\right)\Delta t', \tag{35}$$

As a result of a certain derivation, we get

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$$\frac{\Delta t - \Delta t'}{\Delta t'} = \frac{1}{2}\frac{V^2}{c^2} + Const, \tag{36}$$

In Eq. (35), if both sides are multiplied by c, due to the concept of invariance of the speed of light, then we get Eqs. (37) and (38)

$$c\Delta t = \frac{1}{\sqrt{\left(1 - \frac{V^2}{c^2}\right)}} c\Delta t' \approx \left(1 + \frac{1}{2}\frac{V^2}{c^2} + Const\right) c\Delta t', \tag{37}$$

$$\Delta\lambda = \left(1 + \frac{1}{2}\frac{V^2}{c^2} + Const\right)\Delta\lambda', \tag{38}$$

Where λ is the wavelength in frame K, λ' the wavelength in frame K', we can write Eq. (38) as follows.

$$\frac{\Delta\lambda - \Delta\lambda'}{\Delta\lambda'} = \frac{1}{2}\frac{V^2}{c^2} + Const, \tag{39}$$

If Eq. (38) is divided by c on both sides, we get Eqs. (40) to (43)

$$\frac{\Delta\lambda}{c} = \left(1 + \frac{1}{2}\frac{V^2}{c^2} + Const\right)\frac{\Delta\lambda'}{c}, \tag{40.a}$$

$$\frac{1}{f} = \left(1 + \frac{1}{2}\frac{V^2}{c^2} + Const\right)\frac{1}{f'}, \tag{40.b}$$

$$\frac{f'}{\left(1 + \frac{1}{2}\frac{V^2}{c^2} + Const\right)} = f, \tag{41}$$

$$\left(1 - \frac{1}{2}\frac{V^2}{c^2} - Const\right) f' = f, \tag{42}$$

$$-\frac{f - f'}{f'} = \frac{1}{2}\frac{V^2}{c^2} + Const, \tag{43}$$

As a result, equations (36), (39), and (43) can be written.

$$\frac{\Delta t - \Delta t'}{\Delta t'} = \frac{\Delta\lambda - \Delta\lambda'}{\Delta\lambda'} = -\frac{f - f'}{f'} = \frac{1}{2}\frac{V^2}{c^2} + Const = \frac{1}{2}\frac{V^2}{c^2} + \frac{\Delta U}{c^2}, \tag{44}$$

The *Const* value is calculated by examining speed-related dimensions in Eq.(44). *Const* can be defined as a function $\frac{1}{2}\frac{V^2}{c^2} = \frac{1}{2}\beta^2$. The speed allowed for a GPS satellite moving in an orbit with a height of $R + h$ is $V = \sqrt{\frac{GM}{R+h}}$ with $\frac{GM}{c^2(R+h)} = \frac{14.98\times10^6}{(3\times10^8)^2} = 16.7 \times 10^{-11}$. Where V is the satellite speed in a local Earth-centered inertial (ECI) reference frame, GM_E is the product of the Newtonian gravitational constant G and Earth's mass M, c is the defined speed of light, and ΔU is the effective gravitational potential on the Earth's rotating geoid. Hence, we can write Eq. (44) as Eqs. (45) to (47)

$$\frac{\Delta t - \Delta t'}{\Delta t'} = \frac{\Delta\lambda - \Delta\lambda'}{\Delta\lambda'} = -\frac{f - f'}{f'} = \frac{1}{2}\frac{V^2}{c^2} + \frac{GM}{c^2(R+h)} = \frac{1}{2}\beta^2 + \frac{\Delta U}{c^2}, \tag{45}$$

where GM is 3.98 $\times 10^{14}$ m^3s^{-2} and $R+h$ is 26.7 $\times 10^6$ m. In this study, ΔU is the potential energy due to the gravitational field, with a frequency of

$$f - f' = -err \times f' \tag{46}$$

$$f = (1 - err) \times f' \tag{47}$$

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When we take GPS satellite data, including velocity, V = 3874 m/s, and $\beta = 1.29 \times 10^{-5}$, we get $\frac{\Delta t - \Delta t'}{\Delta t'} = \frac{\Delta\lambda - \Delta\lambda'}{\Delta\lambda'} = -\frac{f - f'}{f'}$. Therefore, the error per day due to time dilation is $\frac{1}{2}\beta^2 + \frac{\Delta U}{c^2} \approx (8.34 + 16.7) \times 10^{-11}$ or 21.6 μs. As a result, at the end of a day, there would be a distance error of $\Delta d = c \times err \times \Delta t' = c \times 21.6\ \mu s = 6.48$ km per day. Eq. (47) explains that the GPS satellite frequency when on Earth must be set at $f = (1 - err)f'$. If the GPS satellite to be orbited has a frequency of $f' = fo$, then the frequency of the receiver stationary on Earth must be $(1 - err)f' = (1 - err)fo$. The magnitude of the general covariant 4-coordinate position vector is

$$x_\mu = g_{\mu\nu}x^\nu = (-ct, x, y, z) = (ct, r, \theta, \varphi) = (-ct, x_1, x_2, x_3) = (-ct, R), \tag{48}$$

The product of the contravariant and covariant 4-position vectors is

$$x^\mu x_\mu = (x^0, x^\mu)(x_0, x_\mu) = (ct, R)(-ct, R) = -c^2t^2 + R^2,$$
$$ds^2 = g_{mn}dx^m dx^n = -c^2dt^2 + dR^2, \tag{49}$$

In Minkowski space-time, the length of a square element is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -c^2dt^2 + dR^2, \tag{50}$$

When we combine the 4-position vectors in the general coordinates of the space-time components in Eq. (48). The resultant 4-position vectors are shown in Eqs. (51) to (54)

$$\boldsymbol{r}' = \boldsymbol{r}_\parallel{}' + \boldsymbol{r}_\perp{}', \tag{51}$$
$$\boldsymbol{r}' = \gamma(\boldsymbol{r}_\parallel - \boldsymbol{V}t) + \boldsymbol{r}_\perp = \gamma(\boldsymbol{r}_\parallel) - \gamma(\boldsymbol{V}t) + \boldsymbol{r}_\perp = \gamma(\boldsymbol{r}_\parallel) - \gamma(\boldsymbol{V}t) + \boldsymbol{r} - \boldsymbol{r}_\parallel, \tag{52}$$

$$\boldsymbol{r}' = \boldsymbol{r} + (\gamma - 1)\boldsymbol{r}_\parallel - \gamma(\boldsymbol{V}t), \tag{53}$$
$$\boldsymbol{r}' = \boldsymbol{r} + (\gamma - 1)\frac{\boldsymbol{r}.\boldsymbol{V}}{V^2}\boldsymbol{V} - \frac{\gamma(ct)}{c}\boldsymbol{V}, \tag{54}$$

Eq. (54) can be expressed as a position tensor as in Eqs. (55) to (56)

$$x^{i'} = x^i + (\gamma - 1)\frac{\boldsymbol{x^j V_j}}{V^2}\boldsymbol{V^i} - \frac{\gamma(x^0)}{c}\boldsymbol{V^i}, \tag{55}$$
$$x^{i'} = x^j\left(\delta^i_j + (\gamma - 1)\frac{V^i V_j}{V^2}\right) - \frac{\gamma x^0}{c}V^i, \tag{56}$$

The 4-velocity vector of the GPS satellite in the K' is written in Eqs. (57) to (60)

$$X' = \frac{dx'}{dt'} = \frac{d\gamma(x - Vt)}{d\gamma\left(t - \frac{Vx}{c^2}\right)} = \frac{d(x - Vt)}{d\left(t - \frac{Vx}{c^2}\right)} = \frac{(dx - Vdt)}{\left(dt - \frac{Vdx}{c^2}\right)} = \frac{(U - V)}{\left(1 - \frac{VU}{c^2}\right)}, \tag{57}$$

$$Y' = \frac{dy'}{dt'} = \frac{dy}{d\left(\gamma\left(t - \frac{Vx}{c^2}\right)\right)} = \frac{dy}{\gamma\left(dt - \frac{Vdx}{c^2}\right)} = \frac{Y}{\gamma\left(1 - \frac{VU}{c^2}\right)}, \tag{58}$$

$$Z' = \frac{dz'}{dt'} = \frac{dz}{d\left(\gamma\left(t - \frac{Vx}{c^2}\right)\right)} = \frac{dz}{\gamma\left(dt - \frac{Vdx}{c^2}\right)} = \frac{Z}{\gamma\left(1 - \frac{VU}{c^2}\right)}, \tag{59}$$

$$c' = c \tag{60}$$

These vectors (X', Y', Z', and c) represent the 4-velocity vectors in directions. x_1, x_2, x_3, as well as the speed of light. Eqs. (57) to (60) can be written as

$$U_\parallel{}' = \frac{(U_\parallel - V)}{\left(1 - \frac{VU_\parallel}{c^2}\right)}, \tag{61}$$

$$U'_\perp = \frac{U_\perp}{\gamma\left(1 - \frac{VU_\parallel}{c^2}\right)}, \tag{62}$$

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We can write the resultant velocity of the 4-velocity vector of the GPS satellite as in Eqs. (63) to (66)

$$U' = U_{\parallel}{}' + U'_{\perp} == \frac{\gamma(U_{\parallel} - V) + U - U_{\parallel}}{\gamma\left(1 - \frac{VU_{\parallel}}{c^2}\right)} = \frac{U + U_{\parallel}(\gamma - 1) - \gamma V}{\gamma\left(1 - \frac{VU_{\parallel}}{c^2}\right)} = \frac{U + U_{\parallel}(\gamma - 1) - \gamma V}{\gamma\left(1 - \frac{VU_{\parallel}}{c^2}\right)}, \quad (63)$$

$$\Gamma'\dot{\boldsymbol{r}}' = \left(\dot{\boldsymbol{r}} + (\gamma - 1)\frac{\dot{\boldsymbol{r}}.\boldsymbol{V}}{V^2}\boldsymbol{V} - \frac{\gamma c}{c}\boldsymbol{V}\right)\Gamma, \quad (64)$$

$$\boldsymbol{v}' = \frac{\left(\dot{\boldsymbol{r}} + (\gamma - 1)\frac{\boldsymbol{v}.\boldsymbol{V}}{V^2}\boldsymbol{V} - \frac{\gamma c}{c}\boldsymbol{V}\right)}{\Gamma'/\Gamma}, \quad (65)$$

$$v^{i'} = \frac{v^j\left(\delta^i_j + (\gamma - 1)\frac{V^i V_j}{V^2}\right) - \frac{\gamma v^0}{c}V^i}{\Gamma'/\Gamma}, \quad (66)$$

Where $\frac{\Gamma'}{\Gamma} = \gamma\left(1 - \frac{VU_{\parallel}}{c^2}\right)$, we can write in Eqs.(67) and (68)

$$\begin{pmatrix} \Gamma' c \\ \Gamma'\dot{x}' \\ \Gamma'\dot{y}' \\ \Gamma'\dot{z}' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma V}{c} & 0 & 0 \\ -\gamma V/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma c \\ \Gamma\dot{x} \\ \Gamma\dot{y} \\ \Gamma\dot{z} \end{pmatrix}, \quad (67)$$

$$\begin{pmatrix} \tilde{v}^0 \\ \tilde{v}^1 \\ \tilde{v}^2 \\ \tilde{v}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma V}{c} & 0 & 0 \\ -\gamma V/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v^0 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}, \quad (68)$$

We can write that the 4-velocity vector of the GPS satellite is $V_\mu = \mathrm{g}_{\mu\nu}\mathrm{V}^{\nu} = (-\Gamma c, \Gamma v)$. The product of the contravariant and covariant 4-velocity vector of the GPS satellite is shown in Eq. (69)

$$\begin{aligned} V^\mu V_\mu = (V^0, V^\mu)(V_0, V_\mu) = (\Gamma c, \Gamma v)(-\Gamma c, \Gamma v) = -\Gamma^2 c^2 - \Gamma^2 vc + \Gamma^2 vc + \Gamma^2 v^2 \\ = -\Gamma^2 c^2 + \Gamma^2 v^2 = -\Gamma^2 c^2\left(1 - \frac{v^2}{c^2}\right) = -c^2, \end{aligned} \quad (69)$$

In this study, we have found a relationship between the error value of a shift in GPS satellite frequency and gamma value, γ, by substituting equation (44) in equations (34) and (35) as follows:

$$\frac{1}{c^2}\frac{V^2}{2} + Const = \frac{1}{2}\frac{V^2}{c^2} + \frac{\Delta U}{c^2} = err \times \Delta t' \quad (70)$$

$$\gamma = \frac{1}{\sqrt{\left(1 - \frac{V^2}{c^2}\right)}} \approx \left(1 + \frac{1}{2}\frac{V^2}{c^2} + Const\right) = 1 + (err \times \Delta t') \quad (71)$$

The Global Positioning System, also called GPS, determines time and position all over the world by using accurate and stable atomic clocks on satellites and on the ground. The gravitational field and the orbital movement of the orbiting satellites cause a frequency shift in these atomic clocks, which cannot be neglected when relativistic effects are considered. According to the findings of the study, there is a shift in time and frequency in the GPS as a result of specific occurrences. We found that Eqs. (36), (39), and (43) can be written in Eq. (44). The results of the research indicate that specific events cause time and frequency shifts in the GPS. We can write that the 4-velocity vector of the GPS satellite is

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$V_\mu = g_{\mu\nu}V^\nu = (-\Gamma c, \Gamma v)$. The product of the contravariant and covariant 4-velocity vector of the GPS satellite can be shown in Eq. (69). As a result of our point of view; readers may find it helpful to learn more about the concept of relativity theory, which may influence the design of GPS systems to ensure maximum accuracy. The findings also provide practical applications for GPS systems and highlight the significance of relativistic effects. Based on our findings, it is feasible to implement the algorithm without complex computations, which makes it more accessible to a wider audience. Furthermore, we explain GPS with common coordinates, making it more straightforward for researchers and scholars to understand.

4. CONCLUSIONS

This study investigated the application of special relativity to the Global Positioning System (GPS), focusing on the impact of relativistic effects on atomic clocks. We demonstrated that gravitational fields and satellite orbital motion induced significant frequency shifts, necessitating corrections to ensure accurate navigation. By employing a novel generalized spacetime coordinate system, we calculated position-4 and velocity-4 vectors to quantify the effects of special relativity on GPS time and frequency. Our analysis, supported by dataset [13], estimated a daily time correction of 21.6 microseconds and a positional imprecision of 6.48 km without relativistic adjustments. These findings confirmed the successful application of special relativity in determining GPS satellite time and frequency.

The developed algorithm simplified relativistic corrections, enhancing computational efficiency and accessibility for broader implementation. This approach not only improves GPS precision but also clarifies the role of relativity in navigation systems. The results underscore the critical influence of relativistic effects on GPS functionality and offer a robust framework for future refinements. This study's insights are valuable for engineers, researchers, and educators seeking to optimize GPS performance and deepen understanding of relativity's practical applications. The methodology remains relevant for advancing navigation technologies and supporting educational efforts in physics and engineering.

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